

# Chapter 4: Higher-Order Differential Equations – Part 1

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# Higher-Order Differential Equations

Most of this chapter deals with **linear** higher-order DE (except 4.10)

In our lecture, we skip 4.10 and focus on  $n$ -th order linear differential equations, where  $n \geq 2$ .

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x) \quad (1)$$

# Methods of Solving Linear Differential Equations

We shall gradually fill up this slide as the lecture proceeds.

- 1 Preliminary: Linear Equations
  - Initial-Value and Boundary-Value Problems
  - Homogeneous Equations
  - Nonhomogeneous Equations
  
- 2 Homogeneous Linear Equations with Constant Coefficients
  - Second Order Equations
  - $n$ -th Order Equations
  
- 3 Summary

# Initial-Value Problem (IVP)

An  $n$ -th order initial-value problem associated with (1) takes the form:

Solve:

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x) \quad (1)$$

subject to:

$$y(x_0) = y_0, y'(x_0) = y_1, \dots, y^{(n-1)}(x_0) = y_{n-1} \quad (2)$$

Here (2) is a set of **initial conditions**.

# Boundary-Value Problem (BVP)

Recall: in Chapter 1, we made 3 remarks on initial/boundary conditions

## Remark (Initial vs. Boundary Conditions)

Initial Conditions: all conditions are at the **same**  $x = x_0$ .

Boundary Conditions: conditions can be at **different**  $x$ .

## Remark (Number of Initial/Boundary Conditions)

“Usually” a  $n$ -th order ODE requires  $n$  initial/boundary conditions to specify an unique solution.

## Remark (Order of the derivatives in the conditions)

Initial/boundary conditions can be the value or the function of 0-th to  $(n - 1)$ -th order derivatives, where  $n$  is the order of the ODE.

# Boundary-Value Problem (BVP)

## Example (Second-Order ODE)

Consider the following second-order ODE

$$a_2(x) \frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x) \quad (3)$$

- IVP: solve (3) s.t.  $y(x_0) = y_0$ ,  $y'(x_0) = y_1$ .
- BVP: solve (3) s.t.  $y(a) = y_0$ ,  $y(b) = y_1$ .
- BVP: solve (3) s.t.  $y'(a) = y_0$ ,  $y(b) = y_1$ .
- BVP: solve (3) s.t. 
$$\begin{cases} \alpha_1 y(a) + \beta_1 y'(a) = \gamma_1 \\ \alpha_2 y(b) + \beta_2 y'(b) = \gamma_2 \end{cases}$$

# Existence and Uniqueness of the Solution to an IVP

Solve

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x) \quad (1)$$

subject to

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \dots, y^{(n-1)}(x_0) = y_{n-1} \quad (2)$$

## Theorem

*If  $a_n(x), a_{n-1}(x), \dots, a_0(x)$  and  $g(x)$  are all continuous on an interval  $I$ ,  $a_n(x) \neq 0$  is not a zero function on  $I$ , and the initial point  $x_0 \in I$ , then the above IVP has a unique solution in  $I$ .*



# Existence and Uniqueness of the Solution to an IVP

Solve

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x) \quad (1)$$

subject to

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \dots, y^{(n-1)}(x_0) = y_{n-1} \quad (2)$$

Throughout this lecture, we assume that on some common interval  $I$ ,

- $a_n(x), a_{n-1}(x), \dots, a_0(x)$  and  $g(x)$  are all continuous
- $a_n(x)$  is not a zero function, that is,  $\exists x \in I$  such that  $a_n(x) \neq 0$ .

# Existence and Uniqueness of the Solution to an BVP

**Note:** Unlike an IVP, even the  $n$ -th order ODE (1) satisfies the conditions in the previous theorem, a BVP corresponding to (1) may have many, one, or no solutions.

## Example

Consider the 2nd-order ODE  $\frac{d^2 y}{dx^2} + y = 0$ , whose general solution takes the form  $y = c_1 \cos x + c_2 \sin x$ . Find the solution(s) to an BVP subject to the following boundary conditions respectively

- $y(0) = 0, y(2\pi) = 0$  Plug it in  $\implies c_1 = 0, c_1 = 0$   
 $\implies c_2$  is arbitrary  $\implies$  infinitely many solutions!
- $y(0) = 0, y(\pi/2) = 0$  Plug it in  $\implies c_1 = 0, c_2 = 0$   
 $\implies c_1 = c_2 = 0 \implies$  a unique solution!
- $y(0) = 0, y(2\pi) = 1$  Plug it in  $\implies c_1 = 0, c_1 = 1$   
 $\implies$  contradiction  $\implies$  no solutions!

## 1 Preliminary: Linear Equations

- Initial-Value and Boundary-Value Problems
- Homogeneous Equations
- Nonhomogeneous Equations

## 2 Homogeneous Linear Equations with Constant Coefficients

- Second Order Equations
- $n$ -th Order Equations

## 3 Summary

# Homogeneous Equation

Linear  $n$ -th order ODE takes the form:

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x) \quad (1)$$

**Homogeneous Equation:**  $g(x)$  in (1) is a zero function:

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = 0 \quad (4)$$

**Nonhomogeneous Equation:**  $g(x)$  in (1) is **not** a zero function. Its *associated homogeneous equation* (4) is the one with the same coefficients except that  $g(x)$  is a zero function

Later in the lecture we will see, when solving a nonhomogeneous equation, we must first solve its associated homogeneous equation (4).

# Differential Operators

We introduce a **differential operator**  $D$ , which simply represent the operation of taking an ordinary differentiation:

## Differential Operator

For a function  $y = f(x)$ , the differential operator  $D$  transforms the function  $f(x)$  to its first-order derivative:  $Dy := \frac{dy}{dx}$ .

Higher-order derivatives can be represented compactly with  $D$  as well:

$$\frac{d^2 y}{dx^2} = D(Dy) =: D^2 y, \quad \frac{d^n y}{dx^n} =: D^n y$$

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y =: \left\{ \sum_{i=0}^n a_i(x) D^i \right\} y$$

# Differential Operators and Linear Differential Equations

**Note:** Polynomials of differential operators are differential operators.

Let  $L := \sum_{i=0}^n a_i(x)D^i$  be an  $n$ -th order differential operator.

Then we can compactly represent the linear differential equation (1) and the homogeneous linear DE (4) as

$$L(y) = g(x), \quad L(y) = 0$$

respectively.

# Linearity and Superposition Principle

$L := \sum_{i=0}^n a_i(x)D^i$  is a **linear operator**: for two functions  $f_1(x), f_2(x)$ ,

$$L(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 L(f_1) + \lambda_2 L(f_2).$$

For any homogeneous linear equation (4), that is,  $L(y) = 0$ , we obtain the following superposition principle.

## Theorem (Superposition Principle: Homogeneous Equations)

Let  $f_1, f_2, \dots, f_k$  be solutions to the homogeneous  $n$ -th order linear equation  $L(y) = 0$  on an interval  $I$ , that is,

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = 0, \quad (4)$$

then the linear combination  $f = \sum_{i=1}^k \lambda_i f_i$  is also a solution to (4).

# Linear Dependence and Independence of Functions

In Linear Algebra, we learned that one can view the collection of all *functions* defined on a common interval as a **vector space**, where linear dependence and independence can be defined respectively.

## Definition (Linear Dependence and Independence)

A set of functions  $\{f_1(x), f_2(x), \dots, f_n(x)\}$  are **linearly dependent** on an interval  $I$  if  $\exists c_1, c_2, \dots, c_n$  not all zero such that

$$c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x) = 0, \quad \forall x \in I,$$

that is, the linear combination is a zero function. If the set of functions is not linearly dependent, it is **linearly independent**.

### Example:

- $f_1(x) = \sin^2 x$ ,  $f_2(x) = \cos^2 x$ ,  $I = (-\pi, \pi)$ : Linearly dependent
- $f_1(x) = 1$ ,  $f_2(x) = x$ ,  $f_3(x) = x^3$ ,  $I = \mathbb{R}$ : Linearly independent.



# Linear Independence of Solutions to (4)

Consider the homogeneous linear  $n$ -th order DE

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x) y = 0, \quad (4)$$

Given  $n$  solutions  $\{f_1(x), f_2(x), \dots, f_n(x)\}$ , we would like to test if they are independent or not.

Of course we can always go back to the definition but it is clumsy...

**Recall:** In Linear Algebra, to test if  $n$  vectors  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  are linearly independent, we can compute the determinant of the matrix

$$\mathbf{V} := [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n].$$

If  $\det \mathbf{V} = 0$ , they are linearly dependent; if  $\det \mathbf{V} \neq 0$ , they are linearly independent.

# Criterion of Linearly Independent Solutions

Consider the homogeneous linear  $n$ -th order DE

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = 0, \quad (4)$$

To test the linear independence of  $n$  solutions  $\{f_1(x), f_2(x), \dots, f_n(x)\}$  to (4), we can use the following theorem.

## Theorem

Let  $\{f_1(x), f_2(x), \dots, f_n(x)\}$  be  $n$  solutions to the homogeneous linear  $n$ -th order DE (4) on an interval  $I$ . They are **linearly independent** on  $I$

$$\iff W(f_1, f_2, \dots, f_n) := \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix} \neq 0.$$

# Fundamental Set of Solutions

We are interested in describing the *solution space*, that is, the subspace spanned by the solutions to the homogeneous linear  $n$ -th order DE

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = 0. \quad (4)$$

How?

**Recall:** In Linear Algebra, we describe a subspace by its *basis*: any vector in the subspace can be represented by a linear combination of the elements in the basis, and these elements are linearly independent.

Similar things can be done here.

## Definition (Fundamental Set of Solutions)

Any set  $\{f_1(x), f_2(x), \dots, f_n(x)\}$  of  $n$  linearly independent solutions to the homogeneous linear  $n$ -th order DE (4) on an interval  $I$  is called a **fundamental set of solutions**.

# General Solutions to Homogeneous Linear DE

**General solution** to an  $n$ -th order ODE:

An  $n$ -parameter family of solutions that can contains *all* solutions.

## Theorem

Let  $\{f_1(x), f_2(x), \dots, f_n(x)\}$  be a fundamental set of solutions to the homogeneous linear  $n$ -th order DE (4) on an interval  $I$ . Then the **general solution** to (4) is

$$y = c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x),$$

where  $\{c_i \mid i = 1, 2, \dots, n\}$  are arbitrary constants.

# Examples

## Example

Consider the DE

$$\frac{d^2 y}{dx^2} = y.$$

Check that both  $y = e^x$  and  $y = e^{-x}$  are solutions to the equation.  
Derive the general solution to the DE.

A: The linear DE is homogeneous.

We see that  $\frac{d^2}{dx^2} e^x = \frac{d}{dx} e^x = e^x$ , and  $\frac{d^2}{dx^2} e^{-x} = \frac{d}{dx} -e^{-x} = e^{-x}$ . Hence they are both solutions to the homogeneous linear second-order DE.

Since

$$\begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -1 - 1 = -2 \neq 0,$$

the two solutions are linearly independent. Hence, the general solution can be written as  $y = c_1 e^x + c_2 e^{-x}$ ,  $c_1, c_2 \in \mathbb{R}$ .

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# General Solutions to Nonhomogeneous Linear DE

Nonhomogeneous linear  $n$ -th order ODE takes the form:

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x) \quad (1)$$

or equivalently,  $L(y) = g(x)$ ,  $L := \sum_{i=0}^n a_i(x) D^i$

where  $g(x)$  is not a zero function.

How to find its general solution?

**Idea:**

- Find the general solution  $y_c$  to the *homogeneous* equation  $L(y) = 0$ .
- Find a solution  $y_p$  to the *nonhomogeneous* equation  $L(y) = g(x)$ .
- The general solution  $y = y_c + y_p$ .

# General Solutions to Nonhomogeneous Linear DE

Nonhomogeneous :

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x) \quad (1)$$

Homogeneous :

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = 0 \quad (4)$$

## Theorem

*Let  $y_p$  be any particular solution to the nonhomogeneous linear  $n$ -th order DE (1) on an interval  $I$ , and  $y_c$  be the general solution to the associated homogeneous linear  $n$ -th order DE (4) on  $I$ , then the general solution to (1) is*

$$y = y_c + y_p.$$



# Proof of the Theorem

**Proof:** Let  $y = f(x)$  be any solution to the nonhomogeneous linear  $n$ -th order DE (1), that is,  $L(y) = g(x)$ .

Now, since both  $y_p$  and  $f$  are solutions to  $L(y) = g(x)$ , we have

$$0 = L(f) - L(y_p) = L(f - y_p).$$

Hence,  $(f - y_p)$  is a solution to the homogeneous linear  $n$ -th order DE (4).

Therefore, any solution to (1) can be represented by the sum of a solution to (4) and the particular solution  $y_p$ .

# Examples

## Example

Consider the DE

$$\frac{d^2 y}{dx^2} = y + 9.$$

Derive the general solution to the DE.

A: The linear DE is nonhomogeneous. The associated homogeneous equation  $\frac{d^2 y}{dx^2} = y$  has the following general solution:

$$y = c_1 e^x + c_2 e^{-x}, \quad c_1, c_2 \in \mathbb{R}.$$

There is an obvious particular solution  $y = -9$ .

Hence, the general solution can be written as

$$y = c_1 e^x + c_2 e^{-x} - 9, \quad c_1, c_2 \in \mathbb{R}$$

# Superposition Principle for Nonhomogeneous Equations

For nonhomogeneous linear differential equations, we have the following superposition principle.

## Theorem (Superposition Principle: Nonhomogeneous Equations)

Let  $f_i(x)$  be a particular solution to the nonhomogeneous  $n$ -th order linear equation  $L(y) = g_i(x)$  on an interval  $I$ , for  $i = 1, 2, \dots, k$ . Then the linear combination  $f = \sum_{i=1}^k \lambda_i f_i$  is a particular solution to the nonhomogeneous  $n$ -th order linear equation

$$L(y) = \sum_{i=1}^k \lambda_i g_i(x).$$

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In this section we focus on solving (that is, giving general solutions to) **Homogeneous Linear Equations with Constant Coefficients**

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = 0, \quad (5)$$

which is a homogeneous linear DE with **constant real coefficients**.

In the textbook, it tells us (without much reasoning) what the form of the general solution should look like, and then we analyze the particular structure of a give equation to derive the exact form.

In this lecture, we try to provide more reasoning, so that you get a clearer big picture.

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# Second Order Equation

We begin with some examples of second order equations.

## Example

Find the general solution of  $\frac{d^2 y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$ .

A: Let us use the differential operator to rewrite this DE as follows:

$$(D^2 - 3D + 2)y = 0.$$

Note that  $L := D^2 - 3D + 2 = (D - 1)(D - 2)$ .

We can view the second-order differential operator  $L$  as a **concatenation of two first-order differential operators**:  $(D - 1)$  and  $(D - 2)$ !

$$y \longrightarrow \boxed{L := D^2 - 3D + 2} \longrightarrow \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y$$

|||

$$L = (D - 2)(D - 1)$$

$$y \longrightarrow \boxed{D - 1} \longrightarrow \boxed{D - 2} \longrightarrow \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y$$

|||

$$L = (D - 1)(D - 2)$$

$$y \longrightarrow \boxed{D - 2} \longrightarrow \boxed{D - 1} \longrightarrow \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y$$



# Second Order Equation

## Example

Find the general solution of  $\frac{d^2 y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$ .

A: We have found the equivalent forms of the above equation

$$Ly = 0 \quad \equiv \quad (D - 2)\{(D - 1)y\} = 0 \quad \equiv \quad (D - 1)\{(D - 2)y\} = 0$$

where  $L := D^2 - 3D + 2 = (D - 1)(D - 2)$ .

### Observation:

- If  $f_1$  is a solution to  $(D - 1)y = 0$ , it is also a solution to  $Ly = 0$ .  
A solution:  $f_1 = e^x$ .
- If  $f_2$  is a solution to  $(D - 2)y = 0$ , it is also a solution to  $Ly = 0$ .  
A solution:  $f_2 = e^{2x}$ .

# Second Order Equation

## Example

Find the general solution of  $\frac{d^2 y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$ .

A: So far we have found two solutions to  $(D^2 - 3D + 2)y = 0$ :

$$f_1 = e^x, \quad \text{corresponds to } (D - 1)y = 0$$

$$f_2 = e^{2x}, \quad \text{corresponds to } (D - 2)y = 0.$$

$f_1$  and  $f_2$  are linearly independent (**Exercise**: check!) and hence  $\{f_1, f_2\}$  is a fundamental set of solutions.

$\implies$  The general solution:

$$y = c_1 f_1 + c_2 f_2 = \boxed{c_1 e^x + c_2 e^{2x}}, \quad c_1, c_2 \in \mathbb{R}.$$

How we solve  $\frac{d^2 y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$ 

- 1 Use a polynomial of  $D$ ,

$$L := p(D) = D^2 - 3D + 2,$$

to rewrite the DE into the form  $Ly = 0$ .

- 2 Factor  $p(D) = (D - 1)(D - 2)$ .
- 3 Observe that a solution to either  $(D - 1)y = 0$  or  $(D - 2)y = 0$  will be a solution to  $Ly = 0$ .
- 4 Find two solutions  $f_1 = e^x$  and  $f_2 = e^{2x}$ , corresponding to  $(D - 1)y = 0$  and  $(D - 2)y = 0$  respectively.
- 5 Check that  $f_1$  and  $f_2$  are linearly independent, and hence they form a fundamental set of solutions.
- 6 Finally we get the general solution  $y = c_1 e^x + c_2 e^{2x}$ .

# $p(D) = a_2 D^2 + a_1 D + a_0$ Has Two Distinct Real Roots

For a homogeneous linear second order DE with constant coefficients  
 $Ly = 0$ , where (WLOG we assume  $a_2 = 1$ )

$$L := p(D) = a_2 D^2 + a_1 D + a_0 = D^2 + a_1 D + a_0 :$$

## Fact

*If  $p(D)$  has two distinct real roots  $m_1$  and  $m_2$ , then we can use the above mentioned method to get a general solution*

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

What if  $p(D)$  has

- Two repeated real roots, or
- Two conjugate complex roots?

# $p(D)$ Has Two Conjugate Complex Roots $\alpha \pm i\beta$

Suppose  $p(D)$  has two conjugate complex roots

$$m_1 = \alpha + i\beta, \quad m_2 = \alpha - i\beta, \quad \alpha, \beta \in \mathbb{R}.$$

If we slightly extend our discussion to complex-valued DE, it is not hard to see that the previous method works again and we get a general (complex-valued) solution

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}, \quad C_1, C_2 \in \mathbb{C}.$$

Still we need to get back to the real domain ...

So, let's do some further manipulation by using the fact that

$$e^{i\theta} = \cos \theta + i \sin \theta.$$

# $p(D)$ Has Two Conjugate Complex Roots $\alpha \pm i\beta$

The general solution to  $Ly = 0$  where  $L = p(D)$  is

$$\begin{aligned}y &= C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x} = C_1 e^{\alpha x} e^{i\beta x} + C_2 e^{\alpha x} e^{-i\beta x} \\&= C_1 e^{\alpha x} (\cos \beta x + i \sin \beta x) + C_2 e^{\alpha x} (\cos \beta x - i \sin \beta x) \\&= (C_1 + C_2) e^{\alpha x} \cos \beta x + i(C_1 - C_2) e^{\alpha x} \sin \beta x\end{aligned}$$

To get a real-valued solution, there are two choices:

- Pick  $C_1 + C_2 = 1, C_1 - C_2 = 0$ : we get  $y = f_1(x) = e^{\alpha x} \cos \beta x$ .
- Pick  $C_1 + C_2 = 0, C_1 - C_2 = -i$ : we get  $y = f_2(x) = e^{\alpha x} \sin \beta x$ .

Since  $f_1$  and  $f_2$  are linearly independent, the general real-valued solution to  $Ly = 0$  where  $L = p(D)$  is

$$y = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x, \quad c_1, c_2 \in \mathbb{R}.$$

## $p(D)$ Has Two Repeated Real Roots $m$

Suppose  $p(D)$  has two repeated real roots  $m$ , which means that

$$p(D) = (D - m)^2.$$

$$L = (D - m)^2$$

$y \rightarrow \boxed{D - m} \rightarrow \boxed{D - m} \rightarrow \frac{d^2y}{dx^2} - 2m \frac{dy}{dx} + m^2$

From the previous discussion, we see that  $y = f_1(x) = e^{mx}$  is a solution to  $(D - m)y = 0$  and hence it is also a solution to  $(D - m)^2y = 0$ .

**Question:**

How to find another solution  $y = f_2(x)$  so that  $f_1$  and  $f_2$  are linearly independent?

## $p(D)$ Has Two Repeated Real Roots $m$

$f_1(x) = e^{mx}$  is a solution to  $(D - m)^2 y = 0$  because:

$$f_1(x) = e^{mx} \longrightarrow \boxed{D - m} \xrightarrow{0} \boxed{D - m} \longrightarrow 0$$

Why not find some  $f_2(x)$  such that after the first  $\boxed{D - m}$  block, the outcome is  $f_1(x) = e^{mx}$ ?

$$f_2(x) = ? \longrightarrow \boxed{D - m} \xrightarrow{e^{mx}} \boxed{D - m} \longrightarrow 0$$

We only need to solve a first order linear DE!

$$\underbrace{e^{-mx}}_{\text{integrating factor}} \quad f_2(x) = \int \overbrace{\underbrace{e^{-mx}}_{\text{integrating factor}} e^{mx}}^1 dx \implies f_2(x) = xe^{mx}.$$



# $p(D)$ Has Two Repeated Real Roots $m$

We have found two solutions to  $(D - m)^2 y = 0$ :

$$f_1(x) = e^{mx}, \quad f_2(x) = xe^{mx},$$

and they are linearly independent (check!).

Hence the general solution to  $p(D)y = 0$  is

$$y = c_1 e^{mx} + c_2 x e^{mx} = \boxed{(c_1 + c_2 x) e^{mx}}.$$

Summary: Second Order Equation  $a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 = 0$ 

Define the following (quadratic) polynomial

$$p(D) := a_2 D^2 + a_1 D + a_0.$$

Roots of $p(D)$	General Solution
Distinct real roots $m_1, m_2 \in \mathbb{R}$	$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$
Conjugate complex roots $\alpha \pm i\beta$ , $\alpha, \beta \in \mathbb{R}$	$y = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x$
Repeated real roots $m \in \mathbb{R}$	$y = (c_1 + c_2 x) e^{mx}$

## 1 Preliminary: Linear Equations

- Initial-Value and Boundary-Value Problems
- Homogeneous Equations
- Nonhomogeneous Equations

## 2 Homogeneous Linear Equations with Constant Coefficients

- Second Order Equations
- $n$ -th Order Equations

## 3 Summary

# $n$ -th Order Equation $a_n \frac{d^n y}{dx^n} + \cdots + a_1 \frac{dy}{dx} + a_0 = 0$

Define

$$p(D) := a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0 = \sum_{i=0}^n a_i D^i$$

and rewrite the  $n$ -th order equation as  $p(D)y = 0$ .

$p(D)$ : a polynomial of order  $n$  with real-valued coefficients.

- $p(D)$  has  $n$  roots in the complex domain (counting the **multiplicity**)
- Complex roots of  $p(D)$  must appear in conjugate pairs.

**Example:**  $p(D) = (D - 1)^3(D - 2)^1(D^2 - 2D + 2)^2$  is a polynomial of order 8, and has the following roots

1	multiplicity 3
2	multiplicity 1
$1 \pm i$	multiplicity 2 for each.

# Finding the General Solution of $p(D)y = 0$

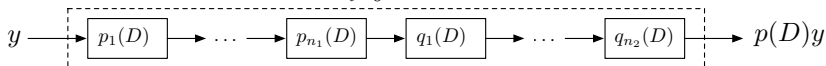
**High-level Idea:** let  $p(D)$  have  $n_1$  distinct real roots  $\{m_i \mid i \in [1 : n_1]\}$ , and  $n_2$  distinct pairs of conjugate complex roots  $\{\alpha_j \pm i\beta_j \mid j \in [1 : n_2]\}$ .

1 Factorize  $p(D) = \sum_{i=0}^n a_i D^i$  as

$$\begin{aligned}
 p(D) &= a_n \left( \prod_{i=1}^{n_1} \overbrace{(D - m_i)^{k_i}}^{p_i(D)} \right) \left( \prod_{j=1}^{n_2} \overbrace{(D^2 - 2\alpha_j D + \alpha_j^2 + \beta_j^2)^{l_j}}^{q_j(D)} \right) \\
 &= a_n \prod_{i=1}^{n_1} p_i(D) \prod_{j=1}^{n_2} q_j(D), \text{ where } n = \sum_{i=1}^{n_1} k_i + 2 \sum_{j=1}^{n_2} l_j.
 \end{aligned}$$

- 2 For each  $i \in [1 : n_1]$ , find  $k_i$  linearly independent solutions of  $p_i(D)y = 0$ .
- 3 For each  $j \in [1 : n_2]$ , find  $2l_j$  linearly independent solutions of  $q_j(D)y = 0$ .
- 4 Combine them all to get  $n$  linearly independent solutions of  $p(D)y = 0$ .

$$p(D) := \sum_{i=0}^n a_i D^i, \quad a_n = 1$$



$$p_i(D) := (D - m_i)^{k_i}, \quad i \in [1 : n_1]; \quad q_j(D) := (D^2 - 2\alpha_j D + \alpha_j^2 + \beta_j^2)^{l_j}, \quad j \in [1 : n_2].$$

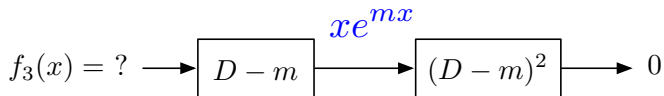
$p(D)$  have  $n_1$  distinct real roots  $\{m_i \mid i \in [1 : n_1]\}$ , and  $n_2$  distinct pairs of conjugate complex roots  $\{\alpha_j \pm i\beta_j \mid j \in [1 : n_2]\}$ .

**Note:** The solutions of different blocks in the above diagram will be linearly independent.

# Solve $(D - m)^k y = 0$

$k = 2$ : two linearly independent solutions  $f_1(x) = e^{mx}$  and  $f_2(x) = xe^{mx}$ .

$k = 3$ : Look at the diagram below:



We only need to solve a first order linear DE!

$$\underbrace{e^{-mx}}_{\text{integrating factor}} f_3(x) = \int \overbrace{\underbrace{e^{-mx}}_{\text{integrating factor}} xe^{mx}}^x dx \implies f_3(x) = x^2 e^{mx} / 2.$$

We can drop the factor of 2 and pick  $f_3(x) = x^2 e^{mx}$ .

Solve  $(D - m)^k y = 0$

$$f_{i+1}(x) = ? \rightarrow \boxed{D - m} \xrightarrow{x^{i-1} e^{mx}} \boxed{(D - m)^i} \rightarrow 0$$

We can repeat this procedure and find  $k$  linearly independent solutions:

$$\boxed{f_1(x) = e^{mx}, f_2(x) = xe^{mx}, f_3(x) = x^2 e^{mx}, \dots, f_k(x) = x^{k-1} e^{mx}}.$$



$$\text{Solve } (D^2 - 2\alpha D + \alpha^2 + \beta^2)^l y = 0$$

$$D^2 - 2\alpha D + \alpha^2 + \beta^2 = (D - m)(D - \bar{m}), \text{ where } m = \alpha + i\beta \in \mathbb{C}.$$

$$\therefore (D^2 - 2\alpha D + \alpha^2 + \beta^2)^l = (D - m)^l (D - \bar{m})^l$$

We can repeat the previous discussion and get  $2l$  linearly independent solutions (in  $\mathbb{C}$ ):

$$F_1(x) = e^{mx}, F_2(x) = xe^{mx}, \dots, F_l(x) = x^{l-1} e^{mx}$$

$$\bar{F}_1(x) = e^{\bar{m}x}, \bar{F}_2(x) = xe^{\bar{m}x}, \dots, \bar{F}_l(x) = x^{l-1} e^{\bar{m}x}$$

For each  $j \in [1 : l]$ , use  $F_j$  and  $\bar{F}_j$  to generate two real-valued solutions:

$$f_{2j-1}(x) = \frac{1}{2}F_j(x) + \frac{1}{2}\bar{F}_j(x) = \operatorname{Re}\{F_j(x)\} = x^{j-1} e^{\alpha x} \cos \beta x$$

$$f_{2j}(x) = \frac{-i}{2}F_j(x) + \frac{i}{2}\bar{F}_j(x) = \operatorname{Im}\{F_j(x)\} = x^{j-1} e^{\alpha x} \sin \beta x.$$

$$\text{Solve } (D^2 - 2\alpha D + \alpha^2 + \beta^2)^l y = 0$$

Here are  $2l$  linearly independent real-valued solutions:

$$\left\{ x^{j-1} e^{\alpha x} \cos \beta x, x^{j-1} e^{\alpha x} \sin \beta x \mid j = 1, 2, \dots, l \right\}$$

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## Short Recap

- Initial-Value Problems (IVP) vs. Boundary-Value Problems (BVP)
- Homogeneous vs Nonhomogeneous Linear ODE
- Fundamental set of solutions and General Solutions
- Linearity and Superposition Principle
- General Solution of Homogeneous Linear Equation with Constant Coefficients – Usage of Polynomial of Differential Operator  $D$

# Self-Practice Exercises

4-1: 1, 9, 13, 17, 21, 25, 35

4-3: 3, 5, 17, 21, 25, 31, 37, 51, 57